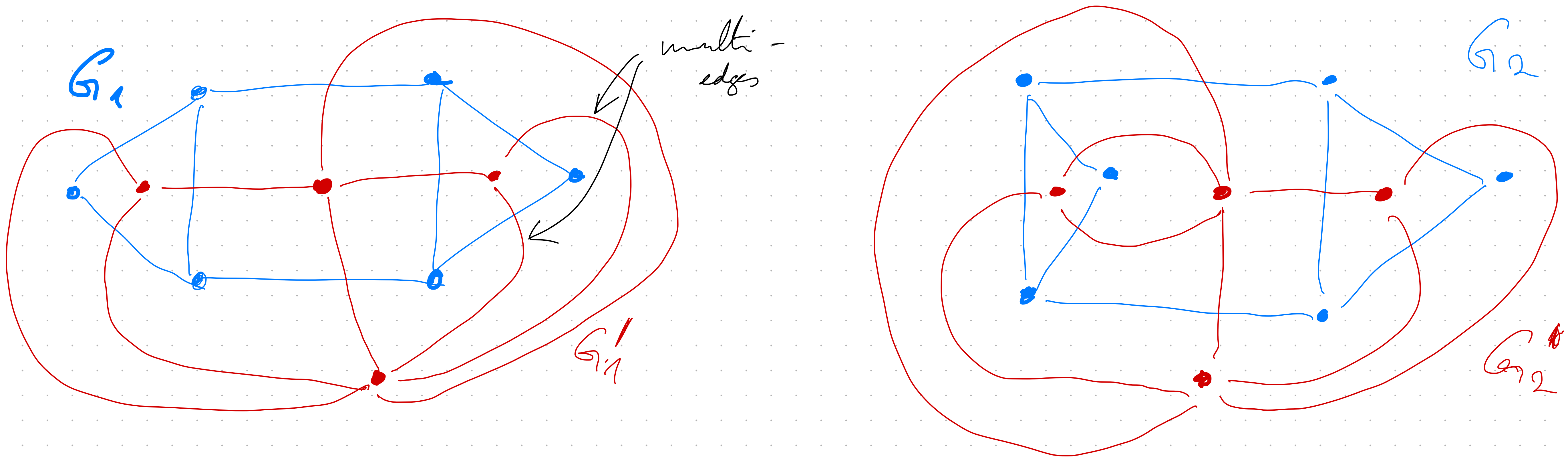
Delaunay Graphs

Delaunay Triangulations / Diagrams / Graphs

Definition (dual graphs):
Let G be a planar, geometric graph, $G = (V, E, F)$, (F also contains the unbounded face).
↑ primal graph

The dual graph G' arises from G by swapping V and F, and for each edge e incident to faces f_1, f_2 G' contains $e' = (f_1', f_2')$.

Example:



Note: • taking the dual can destroy isomorphism!
• $G'' = G$

Definition (triangulation):
Let S = set of pts in $\mathbb{R}^2$.
A triangulation $T(S)$ is a maximal planar graph over S, i.e., no edge can be added without destroying planarity.

Properties:
• Ex always a triangulation (in 2D!)
• Ex only finite number of triangulations
  – Ex $O(59^n)$ many triangulations
• All $T(S)$ have exactly $2n-2-k$ triangles, where k = # pts on the $CH(S)$.
  # edges in $T(S) = 3n-3-k$

- The border of $T(S) = CH(S)$

Def.: Delaunay Triangulation

Let S = set of pts in $\mathbb{R}^2$, and $V(S)$ = Voronoi diagram over S.

Then, denote by $D(S)$ a geometric graph, $D(S) = (S, E_D, F_D)$,

s.t.

$$e = (p, q) \in E_D \;:\Longleftrightarrow\; R(p) \text{ and } R(q) \text{ are neighbors in } V(S).$$

Delaunay edge

In the following, S in general position

here: no 4 pts in S are on a common circle.

Lemma:

Let S = set of pts in general pos.

Then, the following hold:

1. $D(S) = V(S)'$
2. $D(S)$ is a triangulation (b/c of general pos.)
3. $p, q \in S$ form $e = (p,q) \in E_D \iff$ ex circle $C(x)$ through p, q that does <u>not</u> contain any other pt in S.
4. $p, q, r \in S$ form a Δ in $E_D \iff$ ex circle $C(x)$ through p, q, r that does <u>not</u> contain any other pt in S.

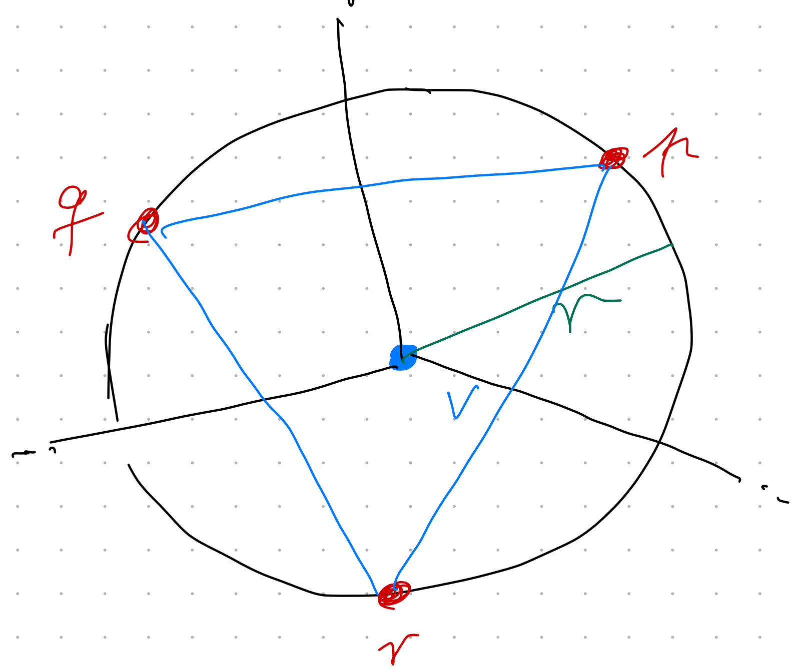
Proof:

1. Obvious from the def. of dual graphs

2. general pos $\Rightarrow$ all Voronoi nodes have $\deg = 3$ $\Rightarrow$ all faces in $V(S)'$ have 3 edges

3. Assume $C(x)$ ex $\Rightarrow$ $x \in B(p,q)$, x lies on edge between $\overline{R(p)} \cap \overline{R(q)}$
$\Rightarrow$ $R(p)$, $R(q)$ share an edge $\Rightarrow$ $(p,q) \in E_D$ by def.
Assume $(p,q) \in E_D$ $\Rightarrow$ $R(p)$, $R(q)$ are neighbors
$\overset{\text{def}}{\Rightarrow}$ $\forall x \in \overline{R(p)} \cap \overline{R(q)}$: $C(x)$ does not contain other pts

4. Consider Voronoi node v $\Rightarrow$ $d(p,v) = d(q,v) = d(r,v) = r$
$\Rightarrow$ $C(v)$ through p,q,r does not contain any other pt in S.
$\Rightarrow$ p,q,r are neighbors
$\Rightarrow$ p,q,r will be connected by Delaunay edges and inside Δpqr there is no other pt.



Assume $\Delta pqr \in F_D \Rightarrow R(p), R(q), R(r)$ are neighbors (pairwise)

$\Rightarrow$ ex Voronoi node v with $d(v,p) = d(v,q) = \ldots$

and $C(v)$ does not contain any other pt

Lemma:

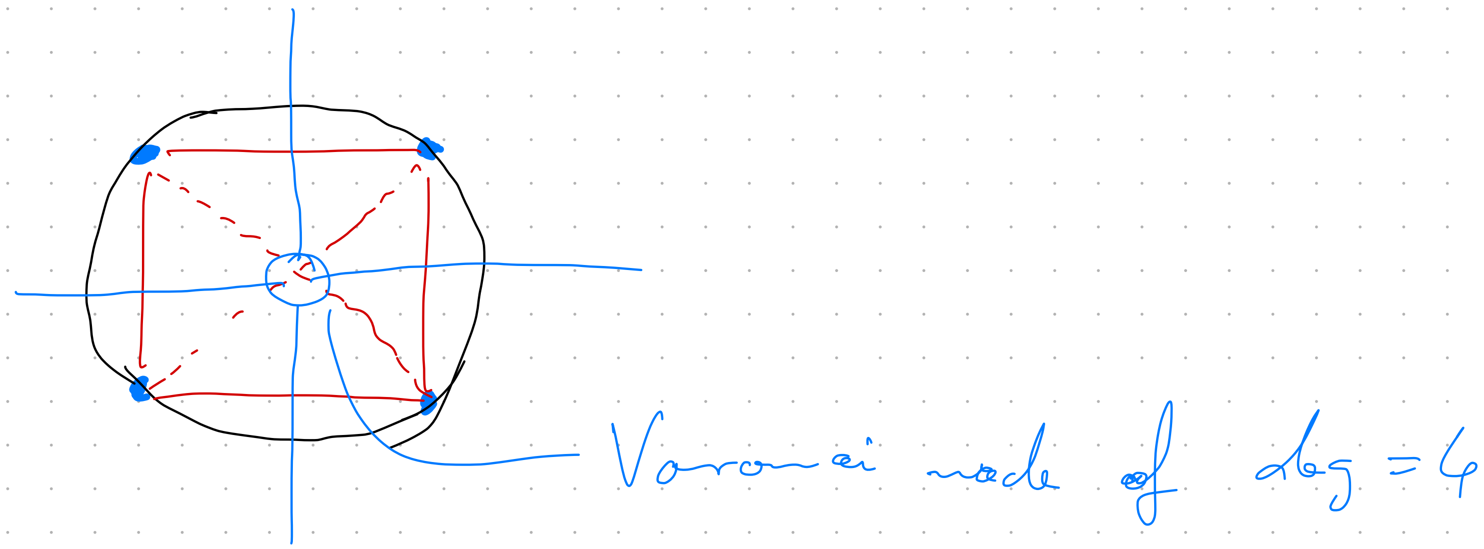
Let S = set pts in general pos.

Then $D(S)$ is unique.

Proof:

Obviously, $V(S)$ is unique, and no V.-node has degree ≥ 3.

Counter example:



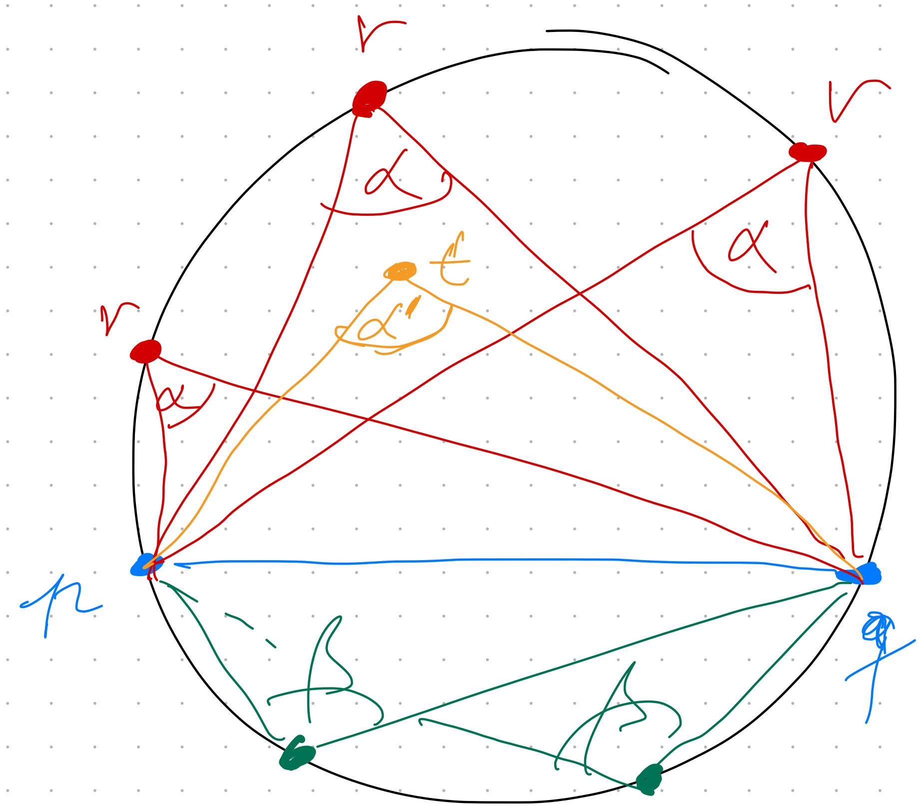
Theorem (Thales):

Let $\overline{pq}$ be a chord in a circle C.

Then inside angle of all pts r on the same segment of C is equal.

(Also: $\alpha + \beta = 180°$)

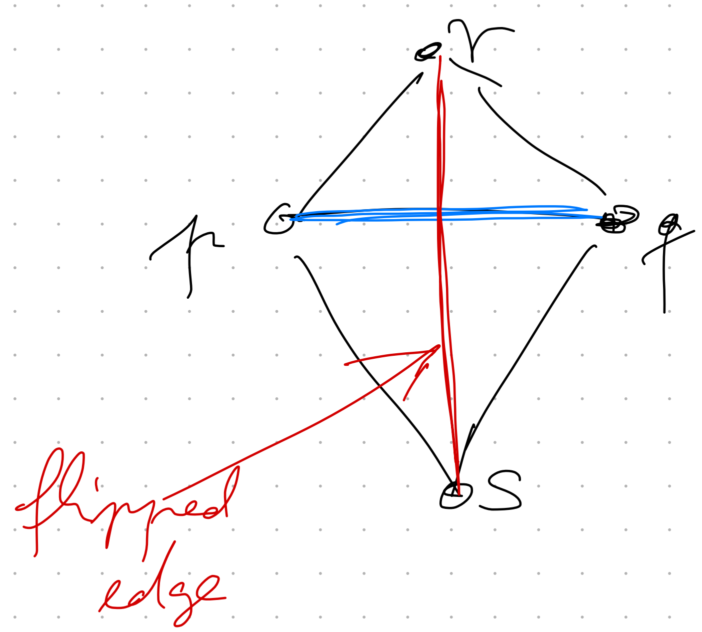
Also, for all t inside C on the "same side" as r: $\alpha_t > \alpha_r$



Notation: $\bigcirc pqr$ = circle through p, q, r
= circumcircle around Δpqr

Operation: edge-flip

Given Δpqr and Δpqs.
Then, $\overline{pq}$ is called "flippable" $\Longleftrightarrow$
p, q, r, s is a convex quad.



Define: angle vector

Let $T(S)$ be triangulation with k Δ's.

Denote with $\alpha(T) = (\alpha_1, \alpha_2, \dots, \alpha_{3k})$ the angle vector over T,
where α's are sorted, i.e., $\alpha_1 \leq \alpha_2 \leq \dots < \alpha_{3k}$.

Def.

$$\alpha(T) > \alpha(T') :\Longleftrightarrow$$

$$\alpha_1 > \alpha_1' \quad \text{or} \quad \alpha_1 = \alpha_1', \dots, \alpha_j = \alpha_j', \quad \alpha_{j+1} > \alpha_{j+1}'.$$

Theorem: (maximal minimal angle)

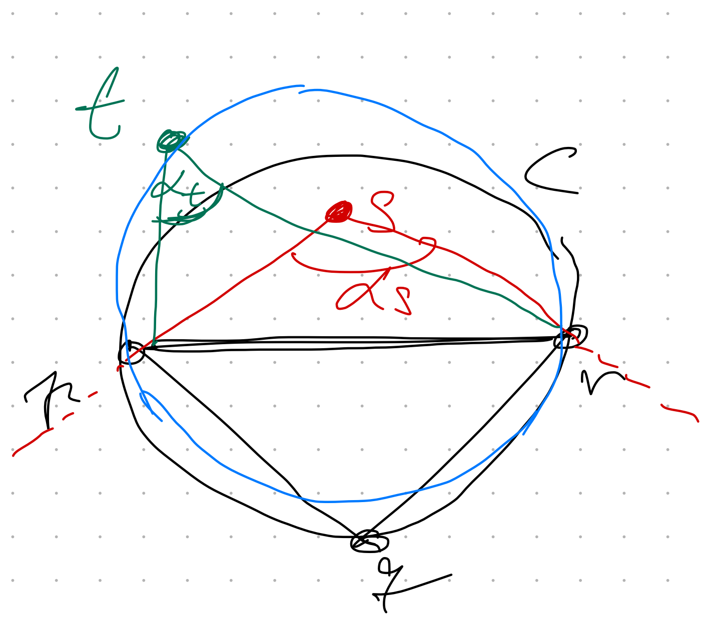
Let S = set of pts in general pos.

Then,

$$\forall\, T(S) : \quad \alpha(D(S)) \geq \alpha(T(S)).$$

In particular,

$$\min_{D(S)} \{\alpha_i\} \geq \min_{T(S)} \{\alpha_j\}$$



Proof:

Assume $T = T(S)$ is not Delaunay

$\Rightarrow$ ex $\triangle pqr$ with pt s inside $\bigcirc pqr$.

Let α_s = angle between tangents of s with $\triangle$

Choose that Non-Delaunay-$\triangle$ with max α_s.

Also, s must lie outside $\triangle pqr$

Claim: $\triangle prs \in T$

Sub-proof: $\underbrace{\text{clearly } \overline{pr} \text{ cannot lie on } CH(S)}$,

Assume $\triangle prs \notin T \Rightarrow \exists t : \triangle prt \in T$

Since $\alpha_s = \max \Rightarrow \alpha_t < \alpha_s \underset{\text{Thales}}{\Rightarrow} t$ outside $\bigcirc pqr$

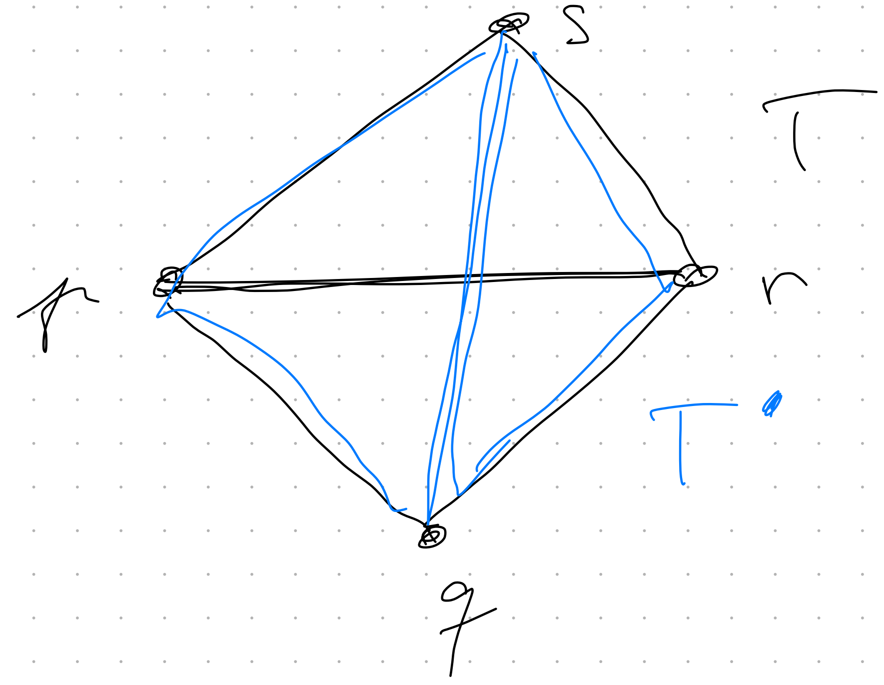
$\Rightarrow s \in$ interior $\bigcirc prt$

$\Rightarrow s$ "sees" $\triangle prt$ under a larger angle $\alpha_s' > \alpha_s$

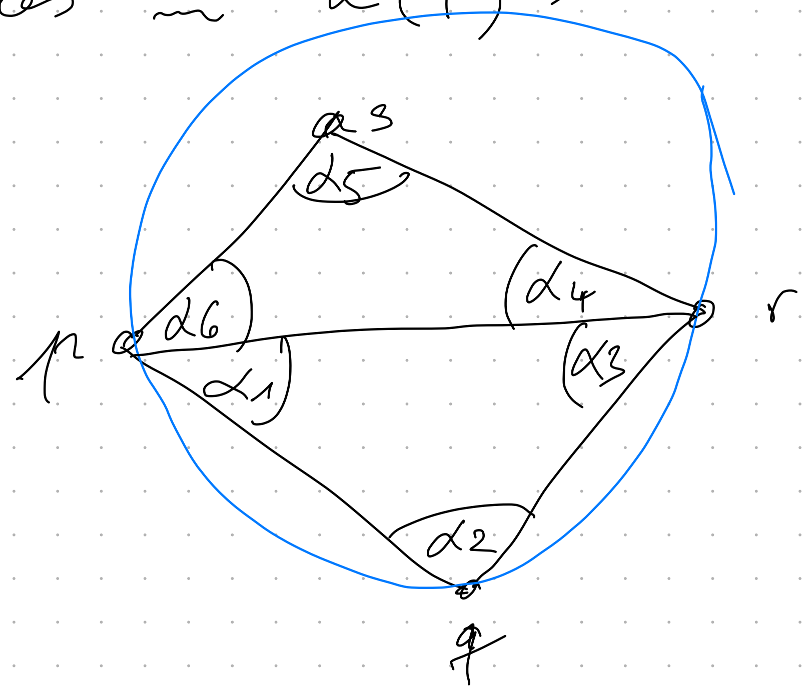
$\Rightarrow$ contradiction to choice of s!

Claim: flipping $\overline{pr}$ leads to a triangulation T'

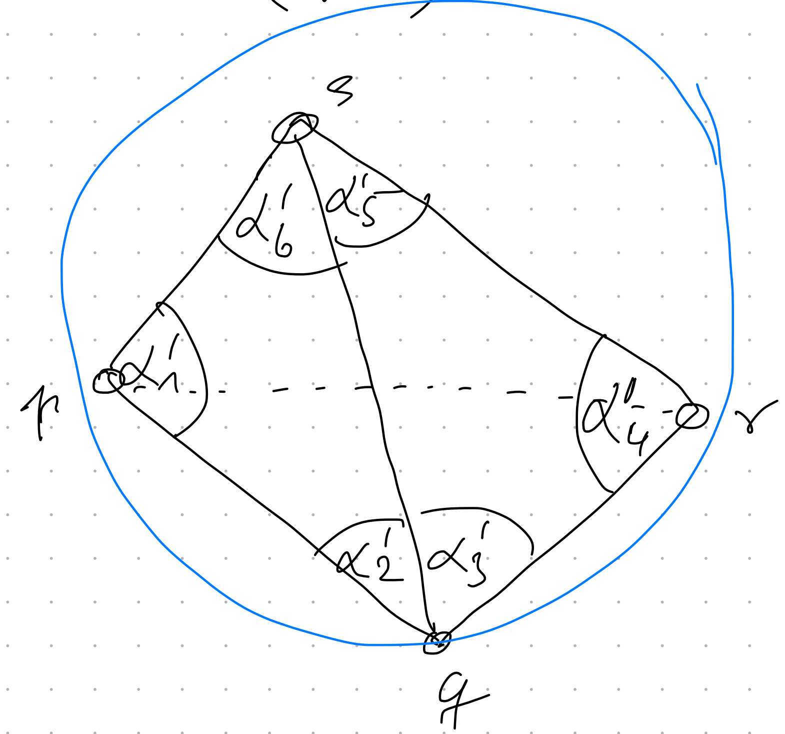
$$\alpha(T') > \alpha(T)$$



Angles in $\alpha(T)$:



in $\alpha(T')$:



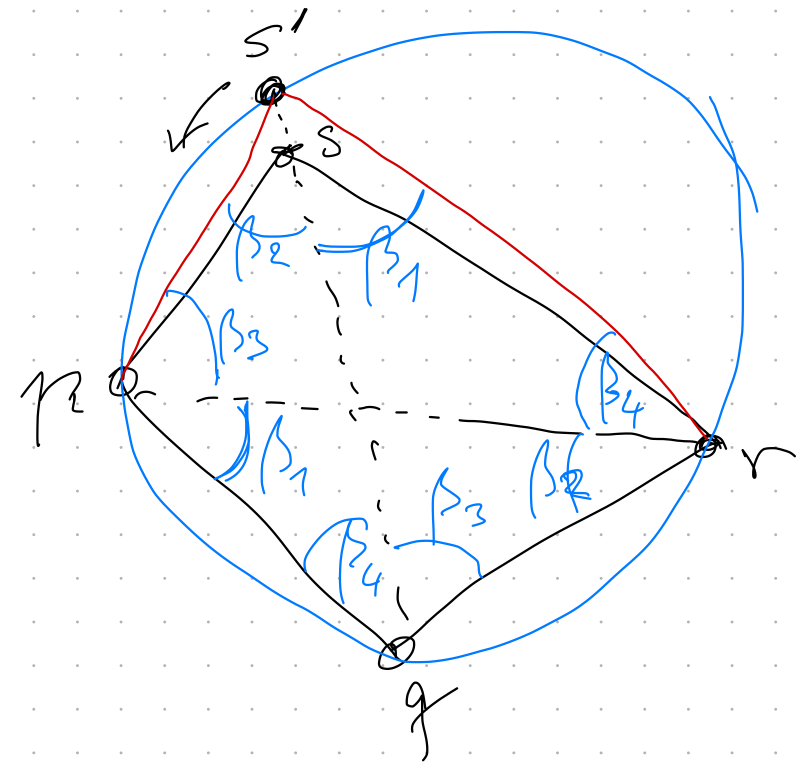
Apply Thales 4x:

$\overline{qr}$, $s' \rightsquigarrow p$ $\rightarrow \beta_1$

$\overline{pq}$, $s' \rightsquigarrow r$ $\rightarrow \beta_2$

$\overline{ps'}$, $r \rightsquigarrow q$ $\rightarrow \beta_4$

$\overline{rs'}$, $p \rightsquigarrow q$ $\rightarrow \beta_3$



From that follows:

$$\left.\begin{aligned} \alpha_1' &= \alpha_1 + \alpha_6 > \alpha_1 \\ \alpha_2' &= \beta_4 > \alpha_4 \\ \alpha_3' &= \beta_3 > \alpha_6 \\ \alpha_4' &= \alpha_3 + \alpha_4 > \alpha_3 \\ \alpha_5' &> \beta_1 = \alpha_1 \\ \alpha_6' &> \beta_2 = \alpha_3 \end{aligned}\right\} \Rightarrow \min\{\alpha_i'\} > \min\{\alpha_i\}$$

$\square$

Other global properties of $D(S)$: (w/o proof)

1. $D(S)$ contains the minimum spanning tree over S as a subgraph

2. $D(S)$ is a geometric spanner with factor $\frac{2\pi}{3\cos\frac{\pi}{6}}$,

i.e. $\forall p, q \in S$: length of path $p \to q \leq \frac{2\pi}{3\cos\frac{\pi}{6}} \cdot \|p-q\|$.

Computing $\mathcal{D}(S)$

Approach: incremental, randomized

Basic op's:

- inserting pt
- in-circle test
- edge flips

Let $D_{i-1} = \mathcal{D}(p_1, \dots, p_{i-1})$

Insert p_i

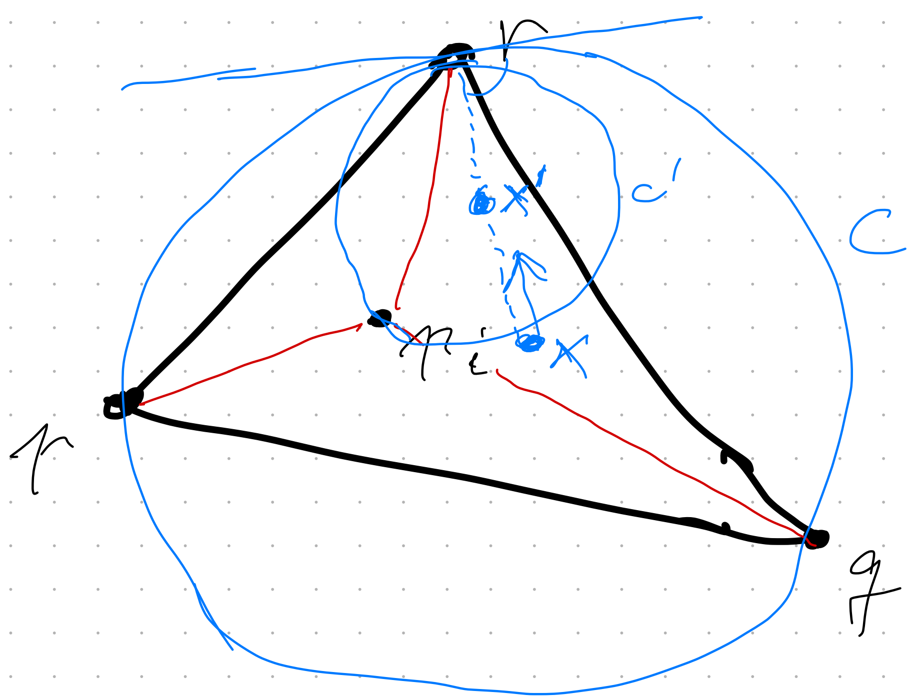
Notation: p_i "is in conflict" with Δpqr $:\Leftrightarrow$ p_i inside $\bigcirc pqr$

Case 1: $p_i \in \Delta pqr$

Claim: $\overline{p_i p}$, $\overline{p_i q}$, $\overline{p_i r}$ are in $\mathcal{D}(p_1, \dots, p_i) = D_i$

Proof: construct circle C touching only p_i, r

$\Rightarrow$ claim



Other triangles:

Assume: p_i in conflict with Δpqr

claim: $\overline{pr}$ will never be a Delaunay again

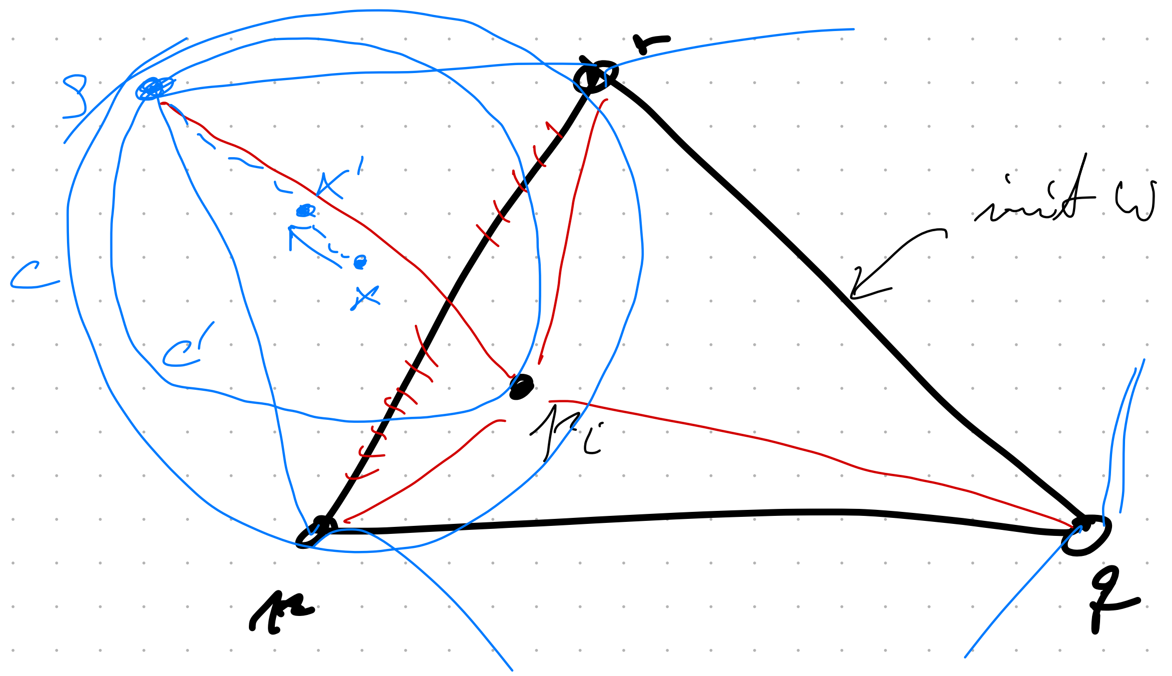
Obvious, since each circle through p, r contains p_i, or s, or both!

Approach: flip $\overline{pr}$

Claim: $\overline{p_i s}$ is Delaunay

Proof: observe $\overline{pr}$ is flippable; only $p_i \in \bigcirc prs$ (b/c D_{i-1} was Delaunay)

construct circle through s, p_i



Approach: define "wavefront" $W \subseteq E_{i-1}$

add p_i → insert $\overline{p_i p}$, $\overline{p_i q}$, $\overline{p_i r}$ into E_i

init $W = \{\overline{pq}, \overline{pr}, \overline{qr}\}$

while $W \neq \emptyset$:

let $e = (p, q) \in W$

let $\triangle pqs$ = tri "opposite" p_i

if e is flippable : // $\Leftrightarrow$ pt "opposite s" is in conflict with $\triangle$

if $p_i \in \bigcirc pqs$:

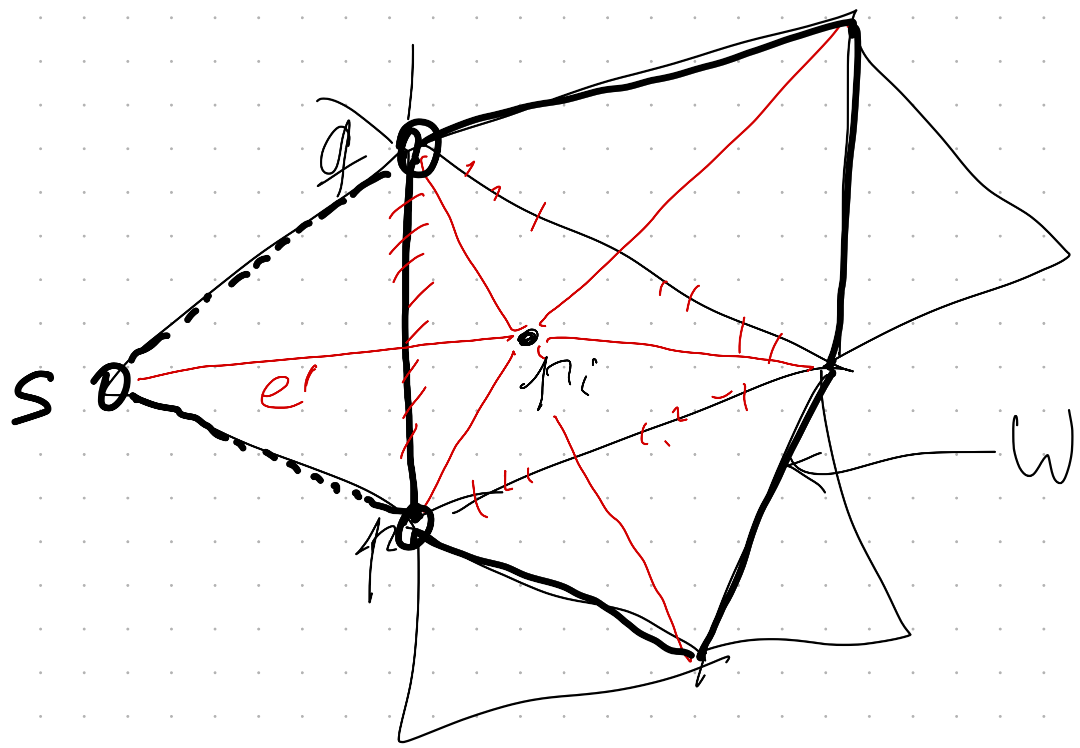
flip $e \to e'$, add e' to E_i

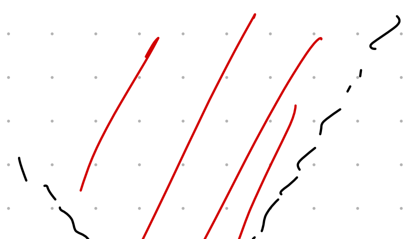
add $\overline{ps}$, $\overline{qs}$ to W

Observe :

if $e = \overline{pq}$ is not flippable

$\Rightarrow$ s cannot be in conflict with $p_i pq$



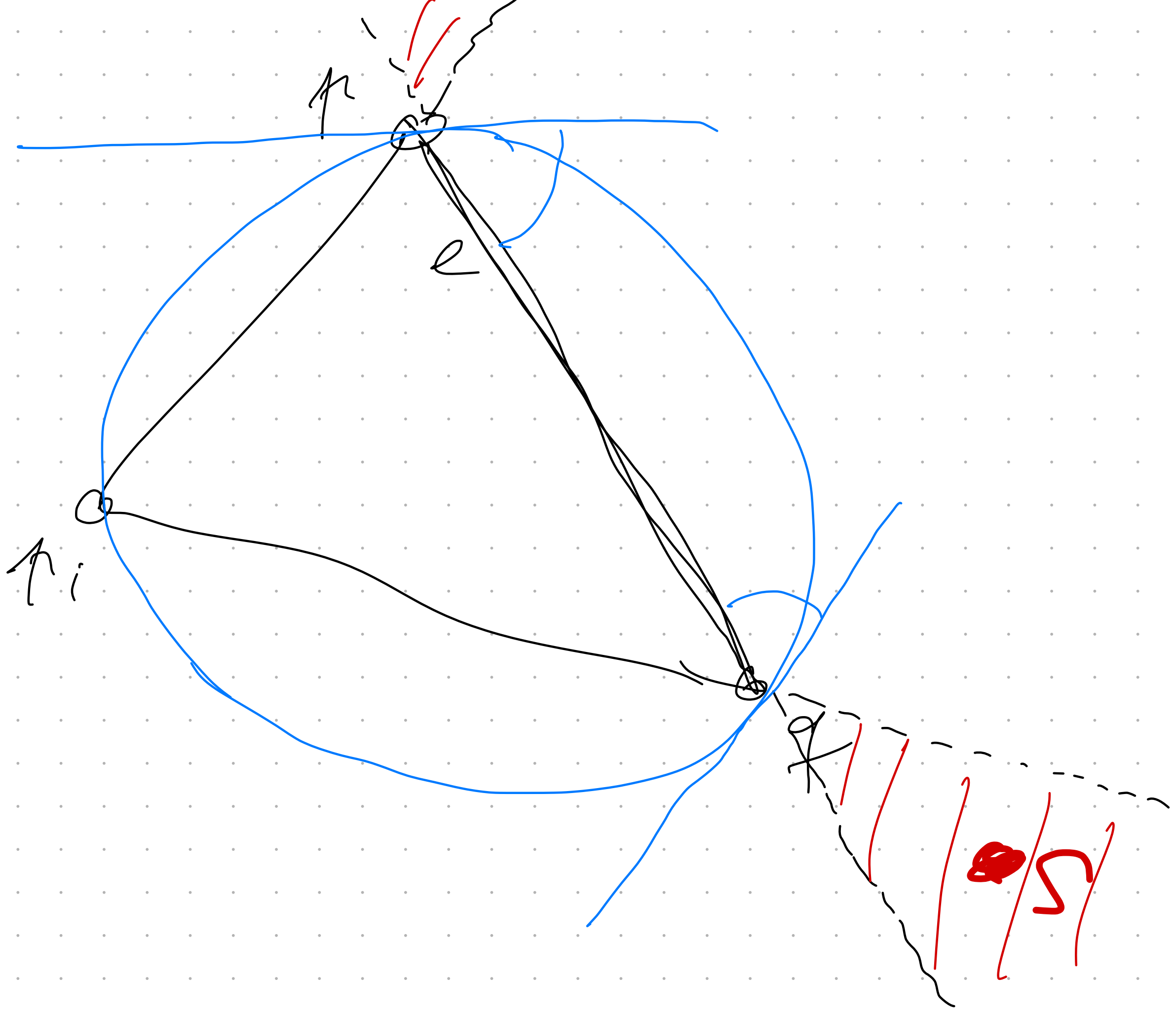


Proof:

e not flippable $\Rightarrow$

p, q, p_i, s not convex

$\Rightarrow$ no matter how small the angles, $s \notin O_{pqp_i}$



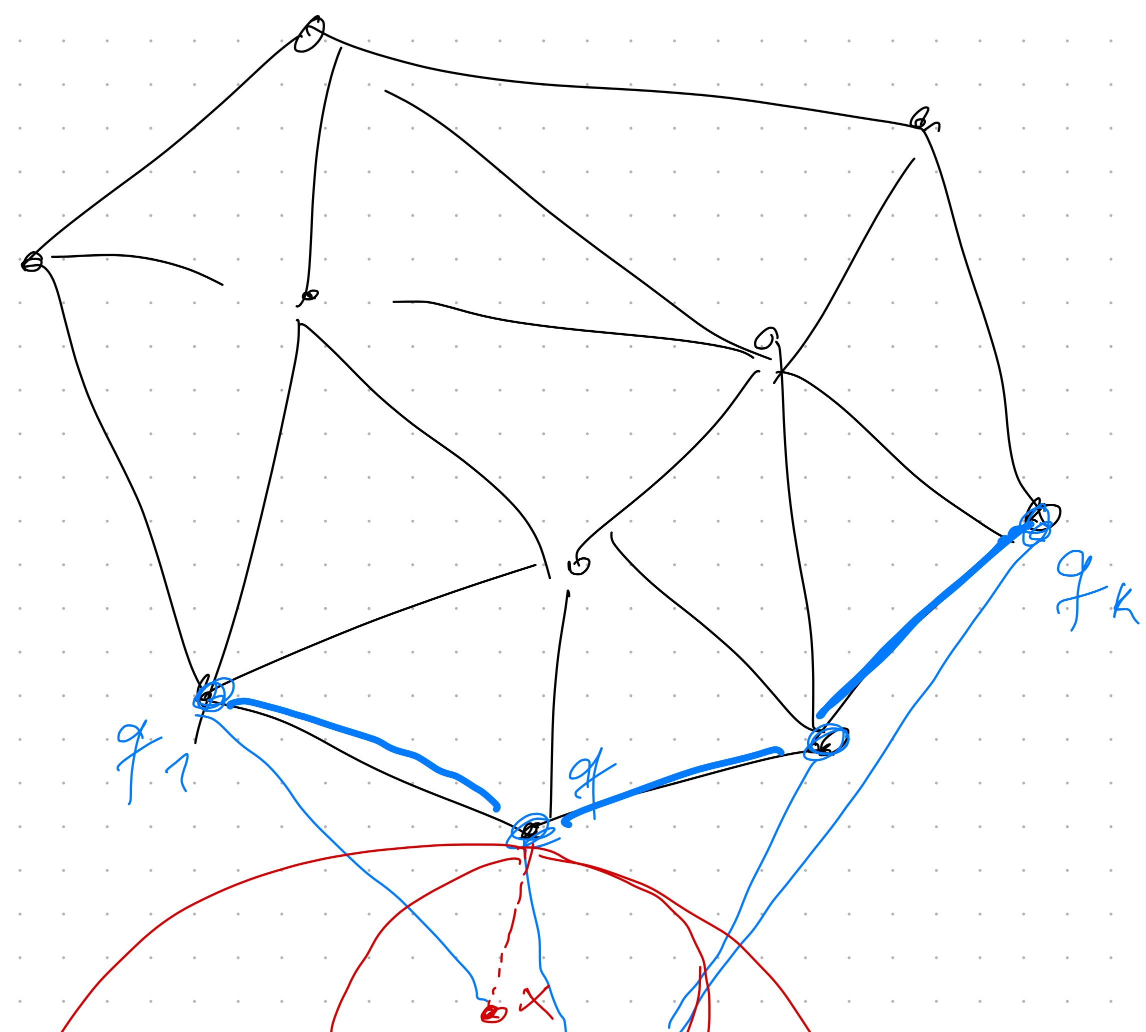
Case 2: $p_i \notin CH(p_1, \dots, p_{i-1})$

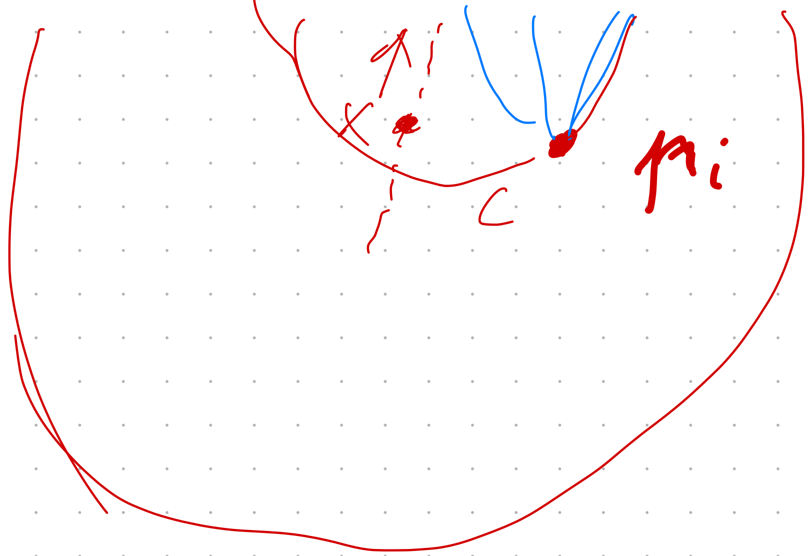
Let $q_1, \dots, q_k \in D_{i-1}$ can be "seen" by p_i

Claim: all $\overline{p_i q_i}$ are Delaunay $\to D_i$

Proof: construct circle

Next: init $W = \{ \overline{q_1 q_2}, \overline{q_2 q_3}, \dots, \overline{q_{k-1} q_k} \}$





Complexity (w/o proof): expected time $O(n \log n)$

On the in-circle test:

Given: p, q, r and s in $\mathbb{R}^2$

Sought: Is s inside $\bigcirc pqr$?

Observe: s inside $\bigcirc pqr \iff \| s - m \|^2 = \ell^2$

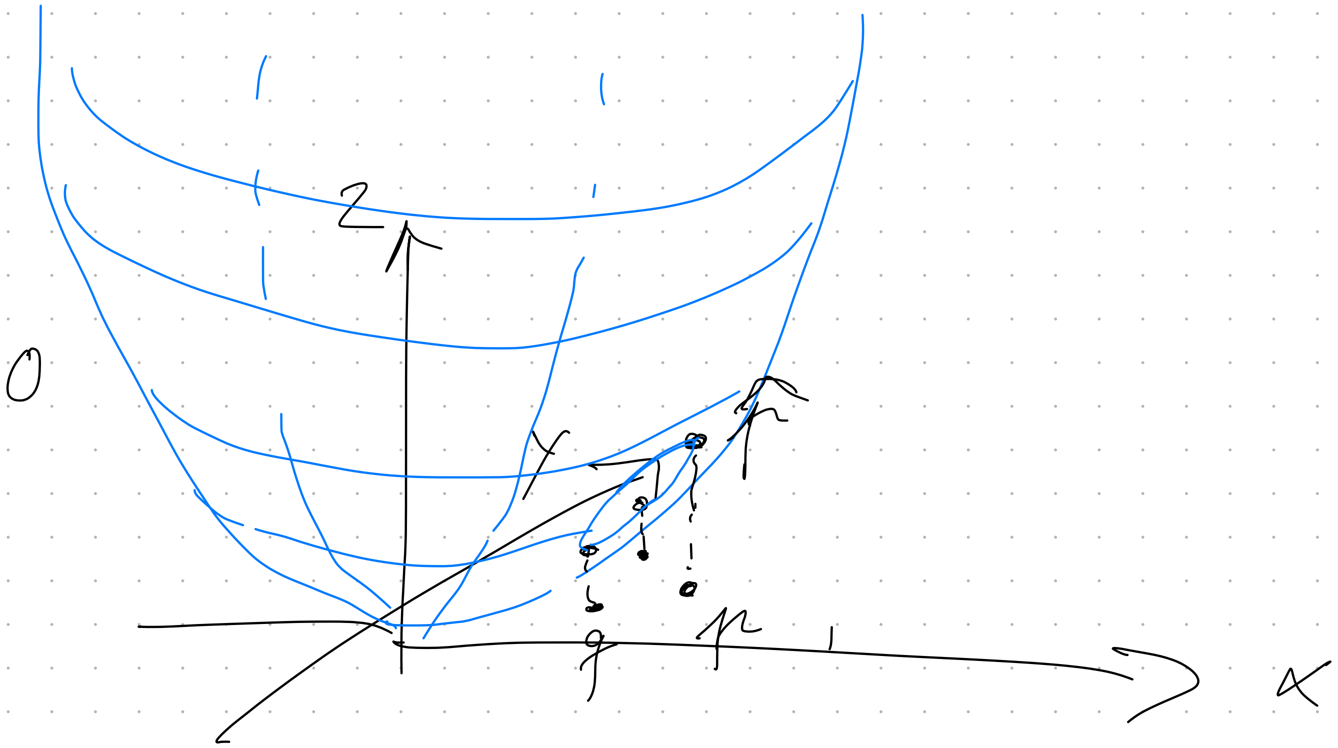
where m, ℓ depend on p, q, r.

Consider projection onto paraboloid: $p \mapsto \hat{p} = (p_x, p_y, p_x^2 + p_y^2)$

Observation (w/o proof):

s on $\bigcirc pqr \Longrightarrow$

$$\hat{s} \cdot \underbrace{\begin{pmatrix} -2m_x \\ -2m_y \\ 1 \end{pmatrix}}_{n = \text{normal}} - \underbrace{m_x^2 + m_y^2 - \ell^2}_{= d} = 0$$



Lemma:

Let p, q, r positively oriented.

Then, s inside $\bigcirc pqr \Longleftrightarrow$

$\hat{s}$ lies below the plane through $\hat{p}, \hat{q}, \hat{r} \Longleftrightarrow$

tetrahedron $\hat{p}, \hat{q}, \hat{r}, \hat{s}$ is negatively oriented $\Longleftrightarrow$

$$\begin{vmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{vmatrix} < 0$$